

On a biharmonic equations with steep potential well and indefinite potential

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Abstract: In this paper, we study the following biharmonic equations:

$$\begin{cases} \Delta^2 u - a_0 \Delta u + (\lambda b(x) + b_0)u = f(u) & \text{in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases} \quad (\mathcal{P}_\lambda)$$

where $N \geq 3$, $a_0, b_0 \in \mathbb{R}$ are two constants, $\lambda > 0$ is a parameter, $b(x) \geq 0$ is a potential well and $f(t) \in C(\mathbb{R})$ is subcritical and superlinear or asymptotically linear at infinity. By the Gagliardo-Nirenberg inequality, we make some observations on the operator $\Delta^2 - a_0 \Delta + \lambda b(x) + b_0$ in $H^2(\mathbb{R}^N)$. Based on these observations, we give a new variational setting to (\mathcal{P}_λ) for $a_0 < 0$. With this new variational setting in hands, we establish some new existence results of the nontrivial solutions to (\mathcal{P}_λ) for all $a_0, b_0 \in \mathbb{R}$ with λ sufficiently large by the variational method. The concentration behavior of the nontrivial solutions as $\lambda \rightarrow +\infty$ is also obtained. It is worth to point out that it seems to be the first time that the nontrivial solution of (\mathcal{P}_λ) is obtained in the case of $a_0 < 0$.

Keywords: Variational method; Biharmonic equation; Potential well; Indefinite problem.

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1 Introduction

In this paper, we study the following biharmonic equations:

$$\begin{cases} \Delta^2 u - a_0 \Delta u + V(x)u = f(x, u) & \text{in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases} \quad (1.1)$$

where $N \geq 3$, $a_0 \in \mathbb{R}$ is a constant and $\lambda > 0$ is a parameter. $V(x)$ and $f(x, u)$ satisfy some conditions to be specified later.

The biharmonic equations in a bounded domain are generally regarded as a mathematical modeling, which can describe some phenomena appeared in physics, engineering and other sciences. For example, the problem of nonlinear oscillation in a suspension bridge [17, 21] and the problem of the static deflection of an elastic plate in a fluid [1]. Due to such applications, the existence and multiplicity of nontrivial solutions for the biharmonic equations in a bounded domain have been extensively studied in the past two decades. We refer the readers to [14, 15, 18, 23, 31] and the references therein. Most of the literatures were devoted to the following Dirichlet-Navier type boundary value problem:

$$\begin{cases} \Delta^2 u - \alpha \Delta u = g(x, u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \quad (\mathcal{P}_\alpha)$$

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where $\Omega \subset \mathbb{R}^N$ is bounded domain with smooth boundary, $\alpha > -\mu_1$ is a parameter and μ_1 is the first eigenvalue of $-\Delta$ in $L^2(\Omega)$. In particular, the existence of sign-changing solutions to (\mathcal{P}_α) was obtained in [18, 23, 32] when $g(x, u)$ is subcritical and superlinear or asymptotically linear at infinity.

In recent years, the study on Problem (1.1), i.e. the biharmonic equations in the whole space \mathbb{R}^N , has begun to attract much attention. We refer the readers to [5, 8, 9, 11, 12, 27, 28, 29] and the references therein. In these literatures, various existence results of the nontrivial solutions to Problem (1.1) were established by the variational method in the case of $a_0 \geq 0$. Indeed, in the case of $a_0 \geq 0$, under some suitable conditions on $V(x)$ and $f(x, u)$, one can give a variational setting to Problem (1.1), as the harmonic equations in the whole space, \mathbb{R}^N in the following Hilbert space

$$\mathcal{V} = \{u \in H^2(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} V^+(x)u^2 dx < +\infty\},$$

where $V^+(x) = \max\{V(x), 0\}$, the inner product and the corresponding norm are respectively given by

$$\langle u, v \rangle_{\mathcal{V}} = \int_{\mathbb{R}^N} (\Delta u \Delta v + a_0 \nabla u \nabla v + V^+(x)uv) dx \quad \text{and} \quad \|u\| = \langle u, u \rangle_{\mathcal{V}}^{\frac{1}{2}}.$$

Thus, the variational method can be used to find the nontrivial solutions of Problem (1.1), see for example [5, 9, 27, 28, 29] and the references therein.

If $a_0 < 0$ then \mathcal{V} with $\langle u, v \rangle_{\mathcal{V}}$ may not be a Hilbert space, since the bilinear operator $\langle u, v \rangle_{\mathcal{V}}$ may not be an inner product in \mathcal{V} for $V(x) \neq 0$ in general. This is quite different from the situation of $V(x) = 0$. Indeed, for example, if we consider the problem (\mathcal{P}_α) in a bounded $\Omega \subset \mathbb{R}^N$, then α can take negative value since the operator $\Delta^2 - \alpha\Delta$ is compact in $L^2(\Omega)$ and the spectrum of $\Delta^2 - \alpha\Delta$ in $L^2(\Omega)$ are the eigenvalues $\{\mu_k^2 + \alpha\mu_k\}$ with the first eigenvalue $\mu_1^2 + \alpha\mu_1$, where $\{\mu_k\}$ are the eigenvalues of $-\Delta$ in $L^2(\Omega)$ with the first eigenvalue $\mu_1 > 0$, so that, if $\alpha > -\mu_1$ then $H_0^1(\Omega) \cap H^2(\Omega)$ is also a Hilbert space with the following inner product

$$\langle u, v \rangle_\alpha = \int_{\Omega} (\Delta u \Delta v + \alpha \nabla u \nabla v) dx,$$

and then one can study (\mathcal{P}_α) by the variational method under some suitable conditions on $g(x, u)$ in the case of $\alpha > -\mu_1$. However, when $V(x) \neq 0$, the operator $\Delta^2 - a_0\Delta + V(x)$ in \mathcal{V} is much more complex due to the potential $V(x)$ and the spectrum of the operator $\Delta^2 - a_0\Delta + V(x)$ in \mathcal{V} is not clear in the case of $a_0 < 0$, also the variational setting of (1.1) is not clear in the case of $a_0 < 0$. Due to these reasons, to our best knowledge, there is few study on Problem (1.1) for the case of $a_0 < 0$. Therefore, a natural question is that does Problem (1.1) have a nontrivial solution for some $a_0 < 0$ and $V(x) \neq 0$? The purpose of this paper is to explore this question.

We assume $V(x) = \lambda b(x) + b_0$, where $b_0 \in \mathbb{R}$ is a constant, $\lambda > 0$ is a parameter and $b(x)$ satisfies the following conditions:

- (B₁) $b(x) \in C(\mathbb{R}^N)$ and $b(x) \geq 0$ on \mathbb{R}^N .
- (B₂) There exists $b_\infty > 0$ such that $|\mathcal{B}_\infty| < +\infty$, where $\mathcal{B}_\infty = \{x \in \mathbb{R}^N \mid b(x) < b_\infty\}$ and $|\mathcal{B}_\infty|$ is the Lebesgue measure of the set \mathcal{B}_∞ .
- (B₃) $\Omega = \text{int}b^{-1}(0)$ is a bounded domain having the smooth boundary $\partial\Omega$ and $\overline{\Omega} = b^{-1}(0)$.

$\lambda b(x)$ is called as the steep potential well for λ sufficiently large under the conditions (B₁)–(B₃) and the depth of the well is controlled by the parameter λ . Such potentials were first introduced by Bartsch and Wang in [3] for the scalar Schrödinger equations. An interesting phenomenon for this kind of Schrödinger equations is that, one can expect to find the solutions which are concentrated at the bottom of the wells as the depth goes to infinity. Due to this interesting property, such topic for the scalar Schrödinger equations was studied extensively in the past decade. We refer the readers to [2, 4, 6, 7, 19, 24, 26] and the references therein. Recently, the steep potential well was also considered for some other elliptic equations and systems, see for example [10, 13, 16, 20, 25, 28, 29, 30, 33] and the references therein. In particular, the steep potential well

was introduced to the biharmonic equations in [20] and was further studied in [28, 29] in the case of $a_0 \geq 0$. For the nonlinearity, we assume that $f(x, t) = f(t)$ and satisfies the following conditions:

$$(F_1) \quad \lim_{t \rightarrow 0} \frac{f(t)}{t} = l_0 \geq 0.$$

$$(F_2) \quad \text{There exists } 2 \leq p < 2^* \text{ such that } \lim_{t \rightarrow \infty} \frac{f(t)}{|t|^{p-2}t} = l_\infty > 0, \text{ where } 2^* = \frac{2N}{N-2}.$$

$$(F_3) \quad \frac{f(t)}{|t|} \text{ is nondecreasing on } \mathbb{R} \setminus \{0\}.$$

$$(F_4) \quad \text{There exists } l_* \in (0, l_\infty] \text{ such that } f(t)t - 2F(t) \geq l_*|t|^p \text{ and } F(t) \geq 0 \text{ for all } t \in \mathbb{R}, \text{ where } F(t) = \int_0^t f(s)ds.$$

Now, under the conditions (B_1) – (B_3) and (F_1) – (F_4) , we mainly study the following problem in this paper:

$$\begin{cases} \Delta^2 u - a_0 \Delta u + (\lambda b(x) + b_0)u = f(u) & \text{in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases} \quad (\mathcal{P}_\lambda)$$

In order to establish a variational framework of (\mathcal{P}_λ) in the case of $a_0 < 0$, we need study the spectrum and Morse index of the operator $\Delta^2 - a_0 \Delta + (\lambda b(x) + b_0)$ in a suitable Hilbert space under the conditions (B_1) – (B_3) . We will borrow some ideas of [7] (see also [33]) to carry on this study. Note that in the case of $a_0 < 0$, the negative part of the operator $\Delta^2 - a_0 \Delta + (\lambda b(x) + b_0)$ is generated by not only $(\lambda b(x) + b_0)^-$ but also $-a_0 \Delta$, where $(\lambda b(x) + b_0)^- = \max\{-(\lambda b(x) + b_0), 0\}$. Therefore, some new ideas and modifications are needed in establishing a variational framework of (\mathcal{P}_λ) in the case of $a_0 < 0$.

Before we state our results, we need to introduce some notations. Let Ω be given in the condition (B_3) and let $\{\mu_k\}$ be the eigenvalues of $-\Delta$ in $L^2(\Omega)$, then it is well known that $0 < \mu_1 < \mu_2 \leq \mu_3 \leq \dots \leq \mu_k < \dots$ with $\mu_k \rightarrow +\infty$ as $k \rightarrow \infty$ and ϕ_k are orthogonal in $L^2(\Omega) \cap H_0^1(\Omega)$ and $\text{span}\{\phi_k\} = H_0^1(\Omega)$, where ϕ_k are the eigenfunctions of μ_k . Since $\partial\Omega$ is smooth due to the condition (B_3) , it is also well known that $\{\phi_k\} \subset C_0^\infty(\bar{\Omega})$. Let H be the Hilbert space $H^2(\Omega) \cap H_0^1(\Omega)$ equipped with the inner product

$$\langle u, v \rangle_H = \int_\Omega (\Delta u \Delta v + \max\{a_0, 0\} \nabla u \nabla v + \max\{b_0, 0\} uv) dx.$$

Then $\text{span}\{\phi_k\} = H$ and ϕ_k are orthogonal in H . We re-denote $\{\mu_k\}$ by $\{\bar{\mu}_n\}$ such that $\bar{\mu}_j < \bar{\mu}_{j+1}$ for all $j \in \mathbb{N}$. Clearly, $\mu_1 = \bar{\mu}_1$. In the case of $\min\{a_0, b_0\} < 0$, we denote

$$k_0^* = \inf \left\{ k \in \mathbb{N} : \frac{\bar{\mu}_k^2 + \max\{a_0, 0\} \bar{\mu}_k + \max\{b_0, 0\}}{\max\{-a_0, 0\} \bar{\mu}_k + \max\{-b_0, 0\}} > 1 \right\} \quad (1.2)$$

and $\bar{\mu}_0 = 0$. Then the main results obtained in this paper can be stated as follows.

Theorem 1.1 *(The superlinear case) Suppose that the conditions (B_1) – (B_3) , (F_1) – (F_2) and (F_4) hold with $p > 2$. If either $\min\{a_0, b_0\} \geq 0$ or $\min\{a_0, b_0\} < 0$ with*

$$\frac{\bar{\mu}_{k_0^*-1}^2 + \max\{a_0, 0\} \bar{\mu}_{k_0^*-1} + \max\{b_0, 0\}}{\max\{-a_0, 0\} \bar{\mu}_{k_0^*-1} + \max\{-b_0, 0\}} < 1$$

then there exist positive constants \bar{l}_0 and $\hat{\Lambda}$ such that Problem (\mathcal{P}_λ) has a nontrivial solution for $\lambda > \hat{\Lambda}$ in the case of $l_0 < \bar{l}_0$.

Theorem 1.2 *(The asymptotically linear case) Suppose that the conditions (B_1) – (B_3) and (F_1) – (F_3) hold with $p = 2$. If either $\min\{a_0, b_0\} \geq 0$ or $\min\{a_0, b_0\} < 0$ with*

$$\frac{\bar{\mu}_{k_0^*-1}^2 + \max\{a_0, 0\} \bar{\mu}_{k_0^*-1} + \max\{b_0, 0\}}{\max\{-a_0, 0\} \bar{\mu}_{k_0^*-1} + \max\{-b_0, 0\}} < 1$$

then there exist positive constants \bar{l}_0 , \bar{l}_∞ and $\hat{\Lambda}$ such that Problem (\mathcal{P}_λ) has a nontrivial solution for $\lambda > \hat{\Lambda}$ in the cases of $l_0 < \bar{l}_0$ and $l_\infty > \bar{l}_\infty$ with $l_\infty \notin \sigma(\Delta^2 - a_0 \Delta + b_0, L^2(\Omega))$, where $\sigma(\Delta^2 - a_0 \Delta + b_0, L^2(\Omega))$ is the spectrum of $\Delta^2 - a_0 \Delta + b_0$ in $L^2(\Omega)$.

Since Problem (\mathcal{P}_λ) depends on the parameter λ , it is natural to investigate the concentration behavior of the solutions for $\lambda \rightarrow +\infty$. Our result in this topic can be stated as follows.

Theorem 1.3 *Suppose that u_λ is the nontrivial solution of Problem (\mathcal{P}_λ) obtained by Theorem 1.1 or Theorem 1.2. Then $u_\lambda \rightarrow u_*$ strongly in $H^2(\mathbb{R}^N)$ as $\lambda \rightarrow +\infty$ up to a subsequence for some $u_* \in H_0^1(\Omega) \cap H^2(\Omega)$. Furthermore, u_* is a nontrivial weak solution of the following equation*

$$\begin{cases} \Delta^2 u - a_0 \Delta u + b_0 u = f(u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.3)$$

Remark 1.1 Theorem 1.3 actually gives an existence result to (1.3) in the case of $a_0 \leq -\mu_1$, where the nonlinearities are superlinear and subcritical or asymptotically linear at infinity. To our best knowledge, such result has not been obtained in literatures no matter what the nonlinearities are superlinear and subcritical or asymptotically linear at infinity.

Through this paper, C and C' will be indiscriminately used to denote various positive constants, $o_n(1)$ and $o_\lambda(1)$ will always denote the quantities tending towards zero as $n \rightarrow \infty$ and $\lambda \rightarrow +\infty$ respectively.

2 The variational setting

In this section, we will give the variational setting of (\mathcal{P}_λ) . Let

$$E_\lambda = \{u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (\lambda b(x) + b_0)^+ u^2 dx < +\infty\},$$

where $(\lambda b(x) + b_0)^+ = \max\{\lambda b(x) + b_0, 0\}$. Then by the condition (B_1) , E_λ is a Hilbert space equipped with the inner product

$$\langle u, v \rangle_\lambda = \int_{\mathbb{R}^N} (\Delta u \Delta v + \max\{a_0, 0\} \nabla u \nabla v + (\lambda b(x) + b_0)^+ uv) dx$$

for all $\lambda > 0$. The corresponding norm in E_λ is given by $\|u\|_\lambda = \langle u, u \rangle_\lambda^{\frac{1}{2}}$. By the condition (B_2) and the Sobolev inequality and the Hölder inequality, for $\lambda > \max\{0, -\frac{b_0}{b_\infty}\}$, we have

$$\|u\|_{L^2(\mathbb{R}^N)}^2 \leq |\mathcal{B}_\infty|^{\frac{2^*-2}{2^*}} S^{-1} \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 + \frac{1}{\lambda b_\infty + b_0} \|u\|_\lambda^2, \quad (2.1)$$

where $\|\cdot\|_{L^p(\mathbb{R}^N)}$ is the usual norms in $L^p(\mathbb{R}^N)$ for all $p \geq 1$, S is the best Sobolev embedding constants from $D^{1,2}(\mathbb{R}^N)$ to $L^{2^*}(\mathbb{R}^N)$ and given by

$$S = \inf\{\|\nabla u\|_{L^2(\mathbb{R}^N)}^2 : u \in D^{1,2}(\mathbb{R}^N), \|u\|_{L^{2^*}(\mathbb{R}^N)}^2 = 1\}.$$

Let $\mathcal{A}_\infty = |\mathcal{B}_\infty|^{\frac{2^*-2}{2^*}} S^{-1}$. If $a_0 > 0$ then by (2.1), we can see

$$\|u\|_{L^2(\mathbb{R}^N)}^2 \leq (\mathcal{A}_\infty a_0^{-1} + \frac{1}{\lambda b_\infty + b_0}) \|u\|_\lambda^2. \quad (2.2)$$

If $a_0 \leq 0$ then by (2.1) and the Young and the Gagliardo-Nirenberg inequalities, we have

$$\|u\|_{L^2(\mathbb{R}^N)}^2 \leq (4\mathcal{A}_\infty^2 B_0^4 + \frac{2}{\lambda b_\infty + b_0}) \|u\|_\lambda^2, \quad (2.3)$$

where $B_0 > 0$ is the constant in the Gagliardo-Nirenberg inequality:

$$\|\nabla u\|_{L^2(\mathbb{R}^N)} \leq B_0 \|\Delta u\|_{L^2(\mathbb{R}^N)}^{\frac{1}{2}} \|u\|_{L^2(\mathbb{R}^N)}^{\frac{1}{2}} \quad \text{for all } u \in H^2(\mathbb{R}^N).$$

Thus, by (2.2) and (2.3), we get

$$\|u\|_{L^2(\mathbb{R}^N)}^2 \leq C_\lambda \|u\|_\lambda^2, \quad (2.4)$$

where

$$C_\lambda = \begin{cases} \mathcal{A}_\infty a_0^{-1} + \frac{1}{\lambda b_\infty + b_0}, & \text{if } a_0 > 0, \\ 4\mathcal{A}_\infty^2 B_0^4 + \frac{2}{\lambda b_\infty + b_0}, & \text{if } a_0 \leq 0. \end{cases}$$

(2.4), together with the Gagliardo-Nirenberg inequality, implies

$$\|\nabla u\|_{L^2(\mathbb{R}^N)} \leq B_0 \|\Delta u\|_{L^2(\mathbb{R}^N)}^{\frac{1}{2}} \|u\|_{L^2(\mathbb{R}^N)}^{\frac{1}{2}} \leq B_0 C_\lambda^{\frac{1}{4}} \|u\|_\lambda. \quad (2.5)$$

Combining (2.4) and (2.5), we deduce that E_λ is embedded continuously into $H^2(\mathbb{R}^N)$ for $\lambda > \max\{0, \frac{-b_0}{b_\infty}\}$. On the other hand, by (2.4) and the conditions (B_1) – (B_2) , we also have

$$\int_{\mathbb{R}^N} (\lambda b(x) + b_0)^- u^2 dx \leq \max\{0, -b_0\} \|u\|_{L^2(\mathbb{R}^N)}^2 \leq \max\{0, -b_0\} C_\lambda^{\frac{1}{2}} \|u\|_\lambda^2. \quad (2.6)$$

Using (F_1) – (F_3) , we can obtain that $0 \leq F(t) \leq 2l_0|t|^2 + C|t|^p$. It follows from (2.4)–(2.6) and the Sobolev embedding theorem that the functional $\mathcal{E}_\lambda(u) : E_\lambda \rightarrow \mathbb{R}$ given by

$$\mathcal{E}_\lambda(u) = \frac{1}{2} \mathcal{D}_\lambda(u, u) - \int_{\mathbb{R}^N} F(x, u) dx$$

is well defined and it belongs to C^1 for $\lambda > \max\{0, -\frac{b_0}{b_\infty}\}$, where

$$\mathcal{D}_\lambda(u, v) = \langle u, v \rangle_\lambda - \mathcal{G}_\lambda(u, v)$$

with $\mathcal{G}_\lambda(u, v) = \max\{-a_0, 0\} \langle \nabla u, \nabla v \rangle_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} (\lambda b(x) + b_0)^- uv dx$. Furthermore, by using a standard argument and the conditions (F_1) – (F_2) , we can show that $\mathcal{E}_\lambda(u)$ is the corresponding functional of (\mathcal{P}_λ) . In what follows, we will make some further observations on $\mathcal{D}_\lambda(u, u)$.

If $\min\{a_0, b_0\} \geq 0$ then $\mathcal{G}_\lambda(u, v) = 0$, which gives that $\mathcal{D}_\lambda(u, v) = \langle u, v \rangle_\lambda$ for all $(u, v) \in E_\lambda$ and then $\mathcal{D}_\lambda(u, u)$ is definite on E_λ .

If $\min\{a_0, b_0\} < 0$ and let

$$\mathcal{M}_0 = \left\{ u \in H : \int_{\Omega} (\max\{-a_0, 0\} |\nabla u|^2 + \max\{-b_0, 0\} |u|^2) dx = 1 \right\}$$

and

$$\mathcal{N}_j = \text{span} \left\{ \phi_i : \phi_i \text{ is the corresponding function of } \overline{\mu}_j \right\}, \quad (2.7)$$

then it is well known that \mathcal{M}_0 is a natural constraint in H and $\dim(\mathcal{N}_j) < +\infty$ for all $j \in \mathbb{N}$. In particular, $\dim(\mathcal{N}_1) = 1$ and ϕ_1 is positive on Ω . Moreover, let

$$\beta_j^0 = \inf_{(\mathcal{M}^{j-1})^\perp} \int_{\Omega} (|\Delta u|^2 + \max\{a_0, 0\} |\nabla u|^2 + \max\{b_0, 0\} |u|^2) dx, \quad j = 1, 2, \dots, \quad (2.8)$$

where $(\mathcal{M}^{j-1})^\perp = \{u \in \mathcal{M}_0 : \langle u, v \rangle_H = 0 \text{ for all } v \in \bigoplus_{i=1}^{j-1} \mathcal{N}_i\}$, then by $\text{span}\{\phi_k\} = H$ and the orthogonality of $\{\phi_k\}$ in H , we can easily see from the Sobolev embedding theorem, the Gagliardo-Nirenberg inequality and the method of Lagrange multipliers that

$$\beta_j^0 = \frac{\overline{\mu}_j^2 + \max\{a_0, 0\} \overline{\mu}_j + \max\{b_0, 0\}}{\max\{-a_0, 0\} \overline{\mu}_j + \max\{-b_0, 0\}} \quad \text{for all } j \in \mathbb{N}$$

and β_j^0 can be attained by $u \in H$ if and only if $u \in \mathcal{N}_j \cap \mathcal{M}_0$.

Lemma 2.1 Suppose that the conditions (B_1) – (B_3) hold. If

$$\lambda > \Lambda_1 := \frac{(\max\{-a_0, 0\}\beta_1^0)^2 B_0^4 - b_0}{b_\infty},$$

then $\beta_1(\lambda) = \inf_{\mathcal{M}_\lambda} \|u\|_\lambda^2$ can be attained by some $e_1(\lambda) \in \mathcal{M}_\lambda$, where $\mathcal{M}_\lambda = \{u \in E_\lambda : \mathcal{G}_\lambda(u, u) = 1\}$. Moreover, $(e_1(\lambda), \beta_1(\lambda))$ satisfies the following equation

$$\begin{cases} \Delta^2 u - \max\{a_0, 0\}\Delta u + (\lambda b(x) + b_0)^+ u = \beta(\max\{-a_0, 0\}\Delta u + (\lambda b(x) + b_0)^- u), & \text{in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases} \quad (2.9)$$

and $(e_1(\lambda), \beta_1(\lambda)) \rightarrow (\phi_1, \beta_1^0)$ strongly in $H^2(\mathbb{R}^N)$ as $\lambda \rightarrow +\infty$ up to a subsequence.

Proof. We first prove that $\beta_1(\lambda)$ can be attained for $\lambda > \Lambda_1$. Indeed, by the Ekeland principle, there exists $\{u_n\} \subset \mathcal{M}_\lambda$ such that

- (1) $\|u_n\|_\lambda^2 = \beta_1(\lambda) + o_n(1)$;
- (2) $\|v\|_\lambda^2 \geq \|u_n\|_\lambda^2 - \frac{1}{n}\|v - u_n\|_\lambda$ for all $v \in \mathcal{M}_\lambda$.

For each $n \in \mathbb{N}$ and $w \in E_\lambda$, by applying the implicit function theorem in a standard way and noting that the conditions (B_1) – (B_2) , we can see that there exist $\varepsilon_n > 0$ and $t_n(l) \in C^1([-\varepsilon_n, \varepsilon_n])$ with $t_n(0) = 1$ and $t'_n(0) = -\mathcal{G}_\lambda(u_n, w)$ such that $t_n(l)u_n + lw \in \mathcal{M}_\lambda$ for $l \in (0, \varepsilon_n]$. It follows from (2) that

$$\begin{aligned} 0 &\geq \|u_n\|_\lambda^2 - \|t_n(l)u_n + lw\|_\lambda^2 - \frac{1}{n}\|(t_n(l) - 1)u_n + lw\|_\lambda \\ &\geq (1 - t_n(l)^2)\|u_n\|_\lambda^2 - 2t_n(l)l\langle u_n, w \rangle_\lambda - l^2\|w\|_\lambda^2 - \frac{t_n(l) - 1}{n}\|u_n\|_\lambda - \frac{l}{n}\|w\|_\lambda. \end{aligned}$$

Multiplying this inequality with l^{-1} on both side and letting $l \rightarrow 0^+$, we deduce

$$\begin{aligned} 0 &\geq (2\|u_n\|_\lambda^2 + \frac{1}{n}\|u_n\|_\lambda) \int_{\mathbb{R}^N} (\max\{-a_0, 0\}\nabla u_n \nabla w + (\lambda b(x) + b_0)^- u_n w) dx \\ &\quad - 2\langle u_n, w \rangle_\lambda - \frac{1}{n}\|w\|_\lambda \\ &= 2(\beta_1(\lambda)\mathcal{G}_\lambda(u_n, w) - \langle u_n, w \rangle_\lambda) + o_n(1). \end{aligned}$$

Since $w \in E_\lambda$ is arbitrary, we must have

$$o_n(1) = \beta_1(\lambda)\mathcal{G}_\lambda(u_n, w) - \langle u_n, w \rangle_\lambda \quad (2.10)$$

for all $w \in E_\lambda$. Let $J_\lambda(u) = \frac{1}{2}\|u\|_\lambda^2 - \frac{\beta_1(\lambda)}{2}\mathcal{G}_\lambda(u, u)$. Then $J'_\lambda(u_n)w = o_n(1)$ for all $w \in E_\lambda$. In particular, by the choice of $\{u_n\}$, we can also see that $\langle J'_\lambda(u_n), u_n \rangle_{E_\lambda^*, E_\lambda} = o_n(1)$, where E_λ^* is the dual space of E_λ and $\langle \cdot, \cdot \rangle_{E_\lambda^*, E_\lambda}$ is the duality pairing of E_λ^* and E_λ . On the other hand, by (1), $\{u_n\}$ is bounded in E_λ . Thus, without loss of generality, we may assume that $u_n \rightharpoonup e_1(\lambda)$ weakly in E_λ as $n \rightarrow \infty$. Note that $J_\lambda(u)$ is C^2 in E_λ for $\lambda > \max\{0, \frac{b_0}{b_\infty}\}$ due to the conditions (B_1) – (B_2) , we have $J'_\lambda(e_1(\lambda)) = 0$ in E_λ^* . It follows from the Sobolev embedding theorem, the conditions (B_1) – (B_2) and similar arguments used in the proofs of (2.4) and (2.5) that

$$\begin{aligned} o_n(1) &= \langle J'_\lambda(u_n) - J'_\lambda(e_1(\lambda)), u_n - e_1(\lambda) \rangle_{E_\lambda^*, E_\lambda} \\ &= \|u_n - e_1(\lambda)\|_\lambda^2 - \beta_1(\lambda) \left(\mathcal{G}_\lambda(u_n - e_1(\lambda), u_n - e_1(\lambda)) \right) \\ &\geq \left(1 - \frac{\beta_1(\lambda) \max\{-a_0, 0\} B_0^2}{(\lambda b_\infty + b_0)^{\frac{1}{2}}} \right) \|u_n - e_1(\lambda)\|_\lambda^2 + o_n(1). \end{aligned} \quad (2.11)$$

Note that by the condition (B_3) , we get

$$\beta_1(\lambda) \leq \frac{\|\phi_1\|_\lambda^2}{\mathcal{G}_\lambda(\phi_1, \phi_1)} = \beta_1^0. \quad (2.12)$$

It follows from (2.11) that for $\lambda > \Lambda_1$, $u_n \rightarrow e_1(\lambda)$ strongly in E_λ as $n \rightarrow \infty$. Thus, $e_1(\lambda) \in \mathcal{M}_\lambda$ and $\beta_1(\lambda)$ can be attained by $e_1(\lambda)$ for $\lambda > \Lambda_1$. By (2.10), we also see that $(e_1(\lambda), \beta_1(\lambda))$ satisfies (2.9).

To complete proof of this lemma, we shall show that $(e_1(\lambda), \beta_1(\lambda)) \rightarrow (\phi_1, \beta_1^0)$ strongly in $H^2(\mathbb{R}^N)$ as $\lambda \rightarrow +\infty$ up to a subsequence. Indeed, by (2.12), we know that $\|e_1(\lambda)\|_\lambda \leq \beta_1^0$ for all $\lambda > \Lambda_1$, which, together with (2.4) and (2.5), implies that $\{e_1(\lambda)\}$ is bounded in $H^2(\mathbb{R}^N)$. Without loss of generality, we may assume that $e_1(\lambda) \rightharpoonup e_1^*$ weakly in $H^2(\mathbb{R}^N)$ as $\lambda \rightarrow +\infty$. Since $\|e_1(\lambda)\|_\lambda \leq \beta_1^0$ for all $\lambda > \Lambda_1$, by the condition (B_1) and the Fatou lemma, we have

$$0 = \liminf_{\lambda \rightarrow +\infty} \frac{\beta_1^0}{\lambda} \geq \liminf_{\lambda \rightarrow +\infty} \int_{\mathbb{R}^N} (b(x) + \frac{b_0}{\lambda})^+ [e_1(\lambda)]^2 dx \geq \int_{\mathbb{R}^N} b(x) [e_1^*]^2 dx. \quad (2.13)$$

It follows from the condition (B_3) that $e_1^* \in H_0^1(\Omega)$. Thanks to the condition (B_2) and the fact that E_λ is embedded continuously into $H^2(\mathbb{R}^N)$ for $\lambda > \max\{0, \frac{-b_0}{b_\infty}\}$, we can see from (2.13) and the Sobolev embedding theorem that $e_1(\lambda) \rightarrow e_1^*$ strongly in $L^2(\mathbb{R}^N)$ as $\lambda \rightarrow +\infty$, then the Gagliardo-Nirenberg inequality yields that $e_1(\lambda) \rightarrow e_1^*$ strongly in $H^1(\mathbb{R}^N)$ as $\lambda \rightarrow +\infty$. On the other hand, by the conditions (B_1) – (B_3) , we obtain from a variant of the Lebesgue dominated convergence theorem (cf. [22, Theorem 2.2]) and the fact that $e_1(\lambda) \rightarrow e_1^*$ strongly in $H^1(\mathbb{R}^N)$ as $\lambda \rightarrow +\infty$ that $e_1^* \in \mathcal{M}_0$, which, together with the definition of β_1^0 and the conditions (B_1) – (B_3) , deduces

$$\liminf_{\lambda \rightarrow +\infty} \beta_1(\lambda) \geq \int_{\Omega} (|\Delta e_1^*|^2 + \max\{a_0, 0\} |\nabla e_1^*|^2 + \max\{b_0, 0\} |e_1^*|^2) dx \geq \beta_1^0.$$

Thus, we must have $\lim_{\lambda \rightarrow +\infty} \beta_1(\lambda) = \beta_1^0$. Now, thanks to the conditions (B_1) – (B_3) once more, we can see from the weak convergence of $\{e_1(\lambda)\}$ in $H^2(\mathbb{R}^N)$ and the Fatou lemma that

$$\beta_1^0 = \lim_{\lambda \rightarrow +\infty} \|e_1(\lambda)\|_\lambda^2 \geq \int_{\Omega} |\Delta e_1^*|^2 + \max\{a_0, 0\} |\nabla e_1^*|^2 + \max\{b_0, 0\} |e_1^*|^2 dx \geq \beta_1^0. \quad (2.14)$$

Thus $\Delta e_k(\lambda) \rightarrow \Delta e_k^*$ strongly in $L^2(\mathbb{R}^N)$ as $\lambda \rightarrow +\infty$ since $e_1(\lambda) \rightarrow e_1^*$ strongly in $L^2(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ as $\lambda \rightarrow +\infty$, so that $e_1(\lambda) \rightarrow e_1^*$ strongly in $H^2(\mathbb{R}^N)$ as $\lambda \rightarrow +\infty$. Note that $e_1^* \in H$ which attains β_1^0 , we finally have $e_1^* = \phi_1$. \blacksquare

Let $\mathcal{N}_{\lambda,1} = \text{span}\{u \in \mathcal{M}_\lambda : \|u\|_\lambda^2 = \beta_1(\lambda)\}$. Then by Lemma 2.1, we can see that $e_1(\lambda) \in \mathcal{N}_{\lambda,1}$ for $\lambda > \Lambda_1$. Moreover, we also have the following lemma.

Lemma 2.2 *Suppose that the conditions (B_1) – (B_3) hold. Then there exists $\Lambda_1^* \geq \Lambda_1$ such that $\mathcal{N}_{\lambda,1} = \text{span}\{e_1(\lambda)\}$ for $\lambda > \Lambda_1^*$.*

Proof. Suppose on the contrary that there exists $\{\lambda_n\}$ satisfying $\lambda_n \rightarrow +\infty$ as $n \rightarrow \infty$ such that $\mathcal{N}_{\lambda_n,1} \neq \text{span}\{e_1(\lambda_n)\} \subset H^2(\mathbb{R}^N)$. It follows that there exists $u(\lambda_n) \in \mathcal{N}_{\lambda_n,1}$ satisfying $u(\lambda_n) \notin \text{span}\{e_1(\lambda_n)\}$ for all $n \in \mathbb{N}$. Without loss of generality, we may assume that $\langle u(\lambda_n), e_1(\lambda_n) \rangle_{\lambda_n} = 0$ for all $n \in \mathbb{N}$. Similarity as in the proof of Lemma 2.1, going if necessary to a subsequence, we may get that $u(\lambda_n) = \phi_1 + o_n(1)$ strongly in $H^2(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} (\lambda_n b(x) + b_0)^+ [u(\lambda_n)]^2 dx = \int_{\Omega} \max\{b_0, 0\} \phi_1^2 dx + o_n(1)$, which, together with Lemma 2.1, implies that $\sqrt{(\lambda_n b(x) + b_0)^+} (u(\lambda_n) - e_1(\lambda_n)) = o_n(1)$ in $L^2(\mathbb{R}^N)$. It follows from a variant of the Lebesgue dominated convergence theorem (cf. [22, Theorem 2.2]) that

$$\|e_1(\lambda_n) - u(\lambda_n)\|_{\lambda_n}^2 = o_n(1).$$

Therefore, we have

$$0 < 2\beta_1^0 = \lim_{n \rightarrow \infty} (\|e_1(\lambda_n)\|_{\lambda_n}^2 + \|u(\lambda_n)\|_{\lambda_n}^2) = \lim_{n \rightarrow \infty} \|e_1(\lambda_n) - u(\lambda_n)\|_{\lambda_n}^2 = 0,$$

this is a contradiction. ■

Let

$$\beta_2(\lambda) = \inf_{(\mathcal{M}_\lambda^1)^\perp} \|u\|_\lambda^2,$$

where $(\mathcal{M}_\lambda^1)^\perp = \{u \in \mathcal{M}_\lambda : \langle u, v \rangle_\lambda = 0 \text{ for all } v \in \mathcal{N}_{\lambda,1}\}$. Then it is easy to see that $\beta_2(\lambda) \geq \beta_1(\lambda)$ for $\lambda > \Lambda_1$. Thus, $\beta_2(\lambda)$ is well defined. Furthermore, we have the following lemma.

Lemma 2.3 *Suppose that the conditions (B_1) – (B_3) hold and let β_2^0 be given in (2.8). If*

$$\lambda > \Lambda_2 := \frac{(\max\{-a_0, 0\}\beta_2^0)^2 B_0^4 - b_0}{b_\infty},$$

then $\beta_2(\lambda)$ can be attained for some $e_2(\lambda) \in E_\lambda$. Moreover, $(e_2(\lambda), \beta_2(\lambda))$ satisfies (2.9) and $(e_2(\lambda), \beta_2(\lambda)) \rightarrow (e_2^0, \beta_2^0)$ strongly in $H^2(\mathbb{R}^N)$ as $\lambda \rightarrow +\infty$ up to a subsequence for some $e_2^0 \in \mathcal{N}_2$, where \mathcal{N}_2 is defined in (2.7).

Proof. For the sake of clarity, the proof will be performed through the following three steps.

Step. 1 We prove that $\limsup_{\lambda \rightarrow +\infty} \beta_2(\lambda) \leq \beta_2^0$.

Let $\varphi \in \mathcal{N}_2$. Then $\varphi = \varphi_\lambda^- + \varphi_\lambda^+$, where φ_λ^- and φ_λ^+ are the projections of φ in $\mathcal{N}_{\lambda,1}$ and $(\mathcal{M}_\lambda^1)^\perp$ respectively. It follows from $\mathcal{N}_2 \subset (\mathcal{M}^1)^\perp$ and Lemmas 2.1 and 2.2 that $\lim_{\lambda \rightarrow +\infty} \|\varphi_\lambda^-\|_\lambda^2 = \lim_{\lambda \rightarrow +\infty} \langle \varphi_\lambda^-, \varphi \rangle_\lambda = 0$ up to a subsequence. By (2.5)–(2.6), $\lim_{\lambda \rightarrow +\infty} \mathcal{G}_\lambda(\varphi_\lambda^-, \varphi_\lambda^-) = 0$ up to a subsequence. Now, using the definitions of $\beta_2(\lambda)$ and β_2^0 , the conditions (B_1) – (B_3) and Lemmas 2.1–2.2, we have

$$\begin{aligned} \limsup_{\lambda \rightarrow +\infty} \beta_2(\lambda) &\leq \limsup_{\lambda \rightarrow +\infty} \frac{\|\varphi_\lambda^+\|_\lambda^2}{\mathcal{G}_\lambda(\varphi_\lambda^+, \varphi_\lambda^+)} \\ &= \limsup_{\lambda \rightarrow +\infty} \frac{\|\varphi - \varphi_\lambda^-\|_\lambda^2}{\mathcal{G}_\lambda(\varphi - \varphi_\lambda^-, \varphi - \varphi_\lambda^-)} \\ &= \limsup_{\lambda \rightarrow +\infty} \frac{\|\varphi\|_\lambda^2 - 2\langle \varphi, \varphi_\lambda^- \rangle_\lambda + \|\varphi_\lambda^-\|_\lambda^2}{\mathcal{G}_\lambda(\varphi, \varphi) - 2\mathcal{G}_\lambda(\varphi, \varphi_\lambda^-) + \mathcal{G}_\lambda(\varphi_\lambda^-, \varphi_\lambda^-)} \\ &= \frac{\int_\Omega |\Delta \varphi|^2 + \max\{a_0, 0\}|\nabla \varphi|^2 + \max\{b_0, 0\}|\varphi|^2 dx}{\int_\Omega \max\{-a_0, 0\}|\nabla \varphi|^2 + \max\{-b_0, 0\}|\varphi|^2 dx} \\ &= \beta_2^0. \end{aligned}$$

Step. 2 We prove that for $\lambda > \Lambda_2$, $\beta_2(\lambda)$ can be attained by some $e_2(\lambda) \in H^2(\mathbb{R}^N)$ and $(e_2(\lambda), \beta_2(\lambda))$ satisfies (2.9).

Indeed, by a similar argument used in the proof of Lemma 2.1, we can show that there exists $\{u_n\} \subset (\mathcal{M}_\lambda^1)^\perp$ such that

- (1) $\|u_n\|_\lambda^2 = \beta_2(\lambda) + o_n(1)$;
- (2) $o_n(1) = \beta_2(\lambda)\mathcal{G}_\lambda(u_n, w) - \langle u_n, w \rangle_\lambda$ for all $w \in (\mathcal{M}_\lambda^1)^\perp$.

Clearly, (1) gives the boundedness of $\{u_n\}$ in E_λ , hence we may assume that $u_n \rightharpoonup e_2(\lambda)$ weakly in E_λ as $n \rightarrow \infty$. Since $\{u_n\} \subset (\mathcal{M}_\lambda^1)^\perp$, we must have $e_2(\lambda) \in (\mathcal{M}_\lambda^1)^\perp$. By (2), we obtain

$$0 = \beta_2(\lambda)\mathcal{G}_\lambda(e_2(\lambda), e_2(\lambda)) - \|e_2(\lambda)\|_\lambda^2,$$

which, together with (1) and similar arguments in the proof of (2.11), implies

$$o_n(1) \geq \left(1 - \frac{\beta_2(\lambda) \max\{-a_0, 0\} B_0^2}{(\lambda b_\infty + b_0)^{\frac{1}{2}}}\right) \|u_n - e_2(\lambda)\|_\lambda^2 + o_n(1).$$

It follows from Step. 1 that $u_n \rightarrow e_2(\lambda)$ strongly in E_λ as $n \rightarrow \infty$ for $\lambda > \Lambda_2$. Hence, by (1)–(2), $\beta_2(\lambda)$ is attained by $e_2(\lambda)$ for $\lambda > \Lambda_2$ and $(e_2(\lambda), \beta_2(\lambda))$ satisfies (2.9).

Step. 3 We prove that $(u_2(\lambda), \beta_2(\lambda)) \rightarrow (e_2^0, \beta_2^0)$ strongly in $H^2(\mathbb{R}^N)$ as $\lambda \rightarrow +\infty$ up to a subsequence for some $e_2^0 \in \mathcal{N}_2$.

Indeed, similarly as in the proof of Lemma 2.1, we can show that $(u_2(\lambda), \beta_2(\lambda)) \rightarrow (e_2^0, \beta_2^0)$ strongly in $H^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ as $\lambda \rightarrow +\infty$ up to a subsequence for some $e_2^0 \in H$. It follows from the conditions (B_1) – (B_3) and a variant of the Lebesgue dominated convergence theorem (cf. [22, Theorem 2.2]) that $e_2^0 \in \mathcal{M}_0$. By Step. 1, we see that either $e_2^0 \in \mathcal{N}_1 \cap \mathcal{M}_0$ or $e_2^0 \in \mathcal{N}_2 \cap \mathcal{M}_0$. If $e_2^0 \in \mathcal{N}_1 \cap \mathcal{M}_0$ then by Lemma 2.1, (2.14) and the Hölder inequality, we have

$$0 = \lim_{\lambda \rightarrow +\infty} \langle u_2(\lambda), e_1(\lambda) \rangle_\lambda = \int_{\Omega} (|\Delta \phi_1|^2 + \max\{a_0, 0\} |\nabla \phi_1|^2 + \max\{b_0, 0\} |\phi_1|^2) dx.$$

It is impossible. Thus, we must have $e_2^0 \in \mathcal{N}_2 \cap \mathcal{M}_0$, which implies that $\liminf_{\lambda \rightarrow +\infty} \beta_2(\lambda) \geq \beta_2^0$, this, together with Step. 1, yields that $\lim_{\lambda \rightarrow +\infty} \beta_2(\lambda) = \beta_2^0$ and e_2^0 attains β_2^0 . Now, by a similar argument used in the proof of Lemma 2.1, we get that $u_2(\lambda) \rightarrow e_2^0$ strongly in $H^2(\mathbb{R}^N)$ as $\lambda \rightarrow +\infty$ up to a subsequence. \blacksquare

Let $\mathcal{N}_{\lambda,2} = \text{span}\{u \in \mathcal{M}_\lambda : \|u\|_\lambda^2 = \beta_2(\lambda)\}$. Then by Lemma 2.3, we can see that $e_2(\lambda) \in \mathcal{N}_{\lambda,2}$ for $\lambda > \Lambda_2$. Moreover, we also have the following.

Lemma 2.4 *Suppose that the conditions (B_1) – (B_3) hold. Then there exists $\Lambda_2^* \geq \Lambda_2$ such that $\beta_1(\lambda) < \beta_2(\lambda)$, $\mathcal{N}_{\lambda,1} \perp \mathcal{N}_{\lambda,2}$ and $\dim(\mathcal{N}_{\lambda,2}) \leq \dim(\mathcal{N}_2)$ for $\lambda > \Lambda_2^*$. Here, we say $\mathcal{N}_{\lambda,1} \perp \mathcal{N}_{\lambda,2}$ in the sense that $\langle u, v \rangle_\lambda = 0$ for all $u \in \mathcal{N}_{\lambda,1}$ and $v \in \mathcal{N}_{\lambda,2}$.*

Proof. Let $u(\lambda), v(\lambda) \in \mathcal{N}_{\lambda,2}$ with $u(\lambda) \notin \text{span}\{v(\lambda)\}$. Without loss of generality, we may assume that $\langle u(\lambda), v(\lambda) \rangle_\lambda = 0$. By Lemma 2.3, we can see that $u(\lambda) \rightarrow e'$ and $v(\lambda) \rightarrow e''$ strongly in $H^2(\mathbb{R}^N)$ as $\lambda \rightarrow +\infty$ up to a subsequence for some $e', e'' \in \mathcal{N}_2$. If $e' = e''$, then by a similar argument used in the proof of Lemma 2.2, we can show that $0 < \beta_2^0 = 0$, which is a contradiction. Hence, we must have $\langle e', e'' \rangle_H = 0$ since $\langle u(\lambda), v(\lambda) \rangle_\lambda = 0$. It follows from Lemmas 2.1 and 2.3 that there exists $\Lambda_2^* \geq \Lambda_2$ such that $\beta_1(\lambda) < \beta_2(\lambda)$, $\mathcal{N}_{\lambda,1} \perp \mathcal{N}_{\lambda,2}$ and $\dim(\mathcal{N}_{\lambda,2}) \leq \dim(\mathcal{N}_2)$ for $\lambda > \Lambda_2^*$. \blacksquare

Now, define $\beta_k(\lambda)$ as

$$\beta_k(\lambda) = \inf_{(\mathcal{M}_\lambda^{k-1})^\perp} \|u\|_\lambda^2 \quad (k = 3, 4, \dots),$$

where $(\mathcal{M}_\lambda^{k-1})^\perp = \{u \in \mathcal{M}_\lambda : \langle u, v \rangle_\lambda = 0 \text{ for all } v \in \bigoplus_{i=1}^{k-1} \mathcal{N}_{\lambda,i}\}$ and $\mathcal{N}_{\lambda,i} = \text{span}\{u \in \mathcal{M}_\lambda : \|u\|_\lambda^2 = \beta_i(\lambda)\}$, then by iterating, we can obtain the following lemma.

Lemma 2.5 *Suppose that the conditions (B_1) – (B_3) hold and $k \in \mathbb{N}$ with $k \geq 3$.*

- (1) *If $\lambda > \Lambda_k := \frac{(\max\{-a_0, 0\} \beta_k^0)^2 B_0^4 - b_0}{b_\infty}$, then $\beta_k(\lambda)$ is well defined and can be attained for some $e_k(\lambda) \in E_\lambda$. Moreover, $(e_k(\lambda), \beta_k(\lambda))$ satisfies (2.9) and $(e_k(\lambda), \beta_k(\lambda)) \rightarrow (e_k^0, \beta_k^0)$ strongly in $H^2(\mathbb{R}^N)$ as $\lambda \rightarrow +\infty$ up to a subsequence for some $e_k^0 \in \mathcal{N}_k$.*
- (2) *There exists $\Lambda_k^* \geq \Lambda_k$ such that $\beta_{k-1}(\lambda) < \beta_k(\lambda)$, $\bigoplus_{i=1}^{k-1} \mathcal{N}_{\lambda,i} \perp \mathcal{N}_{\lambda,k}$ and $\dim(\mathcal{N}_{\lambda,k}) \leq \dim(\mathcal{N}_k)$ for $\lambda > \Lambda_k^*$.*

By Lemmas 2.1, 2.3 and 2.5, $\bigoplus_{i=1}^{k_0^*-1} \mathcal{N}_{\lambda,i}$ and $(\mathcal{M}_\lambda^{k_0^*-1})^\perp$ are well defined for $\lambda > \Lambda_{k_0^*}^*$, where k_0^* is given by (1.2).

Lemma 2.6 *Suppose that the conditions (B_1) – (B_3) hold. If $\beta_{k_0^*-1}^0 < 1$, then we have*

- (1) $\mathcal{D}_\lambda(u, u) \leq \left(1 - \frac{1}{\beta_{k_0^*-1}^*(\lambda)}\right) \|u\|_\lambda^2 = \left(1 - \frac{1}{\beta_{k_0^*-1}^0} + o_\lambda(1)\right) \|u\|_\lambda^2$ in $\bigoplus_{i=1}^{k_0^*-1} \mathcal{N}_{\lambda,i}$;
- (2) $\mathcal{D}_\lambda(u, u) \geq \left(1 - \frac{1}{\beta_{k_0^*}^*(\lambda)}\right) \|u\|_\lambda^2 = \left(1 - \frac{1}{\beta_{k_0^*}^0} + o_\lambda(1)\right) \|u\|_\lambda^2$ in $(\mathcal{M}_\lambda^{k_0^*-1})^\perp$.

Proof. The proof follows immediately from Lemmas 2.1, 2.3 and 2.5. \blacksquare

Remark 2.1 By Lemmas 2.2, 2.4 and 2.5, we also have $\bigoplus_{i=1}^{k_0^*-1} \mathcal{N}_{\lambda,i} = \emptyset$ in the case of $\beta_1^0 > 1$ while $\bigoplus_{i=1}^{k_0^*-1} \mathcal{N}_{\lambda,i} \neq \emptyset$ and $\dim(\bigoplus_{i=1}^{k_0^*-1} \mathcal{N}_{\lambda,i}) \leq \dim(\bigoplus_{i=1}^{k_0^*-1} \mathcal{N}_i)$ in the case of $\beta_1^0 < 1$.

3 The existence of nontrivial solutions

We first consider the case of $\min\{a_0, b_0\} < 0$. Due to the decomposition of E_λ in this case, we will obtain the nonzero critical points of $\mathcal{E}_\lambda(u)$ by using the linking method.

Lemma 3.1 *Suppose that the conditions (B_1) – (B_3) and (F_1) – (F_2) hold with $p > 2$ and $\min\{a_0, b_0\} < 0$. If $\beta_{k_0^*-1}^0 < 1$ and $l_0 d_0 < (1 - \frac{1}{\beta_{k_0^*}^0})$ then there exists $\bar{\Lambda}_0 > 0$ such that*

$$\inf_{(\mathcal{M}_\lambda^{k_0^*-1})^\perp \cap \mathbb{B}_{\lambda, \rho_0}} \mathcal{E}_\lambda(u) \geq \kappa_0 \quad \text{and} \quad \sup_{\partial \mathcal{Q}_{\lambda, R_0}} \mathcal{E}_\lambda(u) \leq 0$$

for all $\lambda > \bar{\Lambda}_0$ with some $\kappa_0 > 0$ and $R_0 > \rho_0 > 0$ independent of λ , where $\mathbb{B}_{\lambda, \rho_0} = \{u \in E_\lambda : \|u\|_\lambda = \rho_0\}$, $\mathcal{Q}_{\lambda, R_0} = \{u = v + te_{k_0^*}(\lambda) : v \in \bigoplus_{i=1}^{k_0^*-1} \mathcal{N}_{\lambda,i}, t \geq 0, \|u\|_\lambda \leq R_0\}$ and

$$d_0 = \begin{cases} \mathcal{A}_\infty a_0^{-1}, & \text{if } a_0 > 0, \\ 4\mathcal{A}_\infty^2 B_0^4, & \text{if } a_0 \leq 0. \end{cases}$$

Proof. Since $l_0 d_0 < (1 - \frac{1}{\beta_{k_0^*}^0})$, there exists $\delta > 0$ such that $(1 + \delta)l_0 d_0 < (1 - \frac{1}{\beta_{k_0^*}^0})$. By the conditions (F_1) – (F_2) , we have $|F(u)| \leq \frac{(1+\delta)l_0}{2}|u|^2 + C|u|^{2^*}$ for all $u \in \mathbb{R}$. It follows from the Sobolev inequality, (2.4) and (2.5) that

$$\begin{aligned} \mathcal{E}_\lambda(u) &\geq \frac{1}{2}(\mathcal{D}_\lambda(u, u) - (1 + \delta)l_0 \|u\|_{L^2(\mathbb{R}^N)}^2) - C\|u\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \\ &\geq \frac{1}{2}(\mathcal{D}_\lambda(u, u) - (1 + \delta)l_0 C_\lambda \|u\|_\lambda^2) - CS^{-\frac{2^*}{2}} B_0^{2^*} C_\lambda^{\frac{2^*}{4}} \|u\|_\lambda^{2^*} \\ &= \frac{1}{2}(\mathcal{D}_\lambda(u, u) - (1 + \delta)l_0(d_0 + o_\lambda(1))\|u\|_\lambda^2) - CS^{-\frac{2^*}{2}} B_0^{2^*} (d_0 + o_\lambda(1))^{\frac{2^*}{4}} \|u\|_\lambda^{2^*} \end{aligned}$$

for all $u \in E_\lambda$. Using Lemma 2.6, we get

$$\mathcal{E}_\lambda(u) \geq \frac{1}{2} \left(1 - \frac{1}{\beta_{k_0^*}^0} + o_\lambda(1) - (1 + \delta)l_0 d_0 \right) \|u\|_\lambda^2 - CS^{-\frac{2^*}{2}} B_0^{2^*} (d_0 + o_\lambda(1))^{\frac{2^*}{4}} \|u\|_\lambda^{2^*}$$

for all $u \in (\mathcal{M}_\lambda^{k_0^*-1})^\perp$. Now, by a standard argument, it is easy to check that there exists $\Lambda_1^* > \Lambda_{k_0^*}^*$ such that

$$\inf_{(\mathcal{M}_\lambda^{k_0^*-1})^\perp \cap \mathbb{B}_{\lambda, \rho_0}} \mathcal{E}_\lambda(u) \geq \kappa_0$$

for all $\lambda > \Lambda_1^*$ with some $\kappa_0 > 0$ and $\rho_0 > 0$ independent of λ . It remains to show that there exists a positive constant $R_0(> \rho_0)$ so large that

$$\sup_{\partial \mathcal{Q}_{\lambda, R_0}} \mathcal{E}_\lambda(u) \leq 0.$$

for λ sufficient large. Indeed, let $u_\lambda \in \partial \mathcal{Q}_{\lambda, R}$ be such that $u_\lambda = R\tilde{u}$ with $\tilde{u}_\lambda \in \mathcal{Q}_{\lambda, 1}$, then one of the following two cases must happen:

- (1) $\tilde{u}_\lambda \in \bigoplus_{i=1}^{k_0^*-1} \mathcal{N}_{\lambda,i}$ and $\|\tilde{u}_\lambda\|_\lambda \leq 1$;
- (2) $\tilde{u}_\lambda \in \mathcal{Q}_{\lambda, 1} \setminus \bigoplus_{i=1}^{k_0^*-1} \mathcal{N}_{\lambda,i}$ and $\|\tilde{u}_\lambda\|_\lambda = 1$.

In the case (1), it follows from Lemma 2.6 and the condition (F_4) that $\mathcal{E}_\lambda(R\tilde{u}_\lambda) \leq 0$ for all $R \geq 0$ and $\lambda > \Lambda_1^*$. In the case (2), also by using Lemma 2.6, we deduce

$$\begin{aligned}\mathcal{E}_\lambda(R\tilde{u}_\lambda) &= R^2 \left(\frac{1}{2} \mathcal{D}_\lambda(\tilde{u}_\lambda, \tilde{u}_\lambda) - \int_{\mathbb{R}^N} \frac{F(x, R\tilde{u}_\lambda)}{(R\tilde{u}_\lambda)^2} \tilde{u}_\lambda^2 dx \right) \\ &\leq R^2 \left(\frac{1}{2} \left(1 - \frac{1}{\beta_{k_0^*}}\right) + o_\lambda(1) - \int_{\mathbb{R}^N} \frac{F(x, R\tilde{u}_\lambda)}{(R\tilde{u}_\lambda)^2} \tilde{u}_\lambda^2 dx \right).\end{aligned}\quad (3.1)$$

On the other hand, since $\tilde{u}_\lambda \in \bigoplus_{i=1}^{k_0^*} \mathcal{N}_{\lambda,i}$, by Lemmas 2.1, 2.3 and 2.5, we have $\tilde{u}_\lambda = \tilde{u} + o_\lambda(1)$ strongly in $H^2(\mathbb{R}^N)$ for some $\tilde{u} \in \bigoplus_{i=1}^{k_0^*} \mathcal{N}_i$ with

$$\int_{\Omega} |\Delta \tilde{u}|^2 + \max\{a_0, 0\} |\nabla \tilde{u}|^2 + \max\{b_0, 0\} |\tilde{u}|^2 dx = 1,$$

which together with the conditions (F_1) – (F_2) gives that

$$\int_{\mathbb{R}^N} \frac{F(x, R\tilde{u}_\lambda)}{(R\tilde{u}_\lambda)^2} \tilde{u}_\lambda^2 dx = \int_{\mathbb{R}^N} \frac{F(x, R\tilde{u})}{(R\tilde{u})^2} \tilde{u}^2 dx + o_\lambda(1).\quad (3.2)$$

Clearly,

$$\|u\|_{\Omega,0} = \left(\int_{\Omega} (|\Delta u|^2 + \max\{a_0, 0\} |\nabla u|^2 + \max\{b_0, 0\} |u|^2) dx \right)^{\frac{1}{2}}$$

is a norm in H , and note that $\dim(\bigoplus_{i=1}^{k_0^*} \mathcal{N}_i) < +\infty$, we see that there exists $d_* > 0$ such that

$$\|u\|_{\Omega,0} \leq d_* \|u\|_{L^2(\mathbb{R}^N)} \quad \text{for all } u \in \bigoplus_{i=1}^{k_0^*} \mathcal{N}_i.\quad (3.3)$$

Now, by the condition (F_2) and the Fatou lemma, there exists $R_0 > \rho_0$ independent of λ such that

$$\int_{\mathbb{R}^N} \frac{F(x, R_0 \tilde{u})}{R_0^2 \tilde{u}^2} \tilde{u}^2 dx \geq \left(1 - \frac{1}{\beta_{k_0^*}}\right).$$

It follows from (3.1) and (3.2) that there exist $\bar{\Lambda}_0 > \Lambda_1^*$ and such that $\sup_{\partial \mathcal{Q}_{\lambda, R_0}} \mathcal{E}_\lambda(u) \leq 0$ for all $\lambda > \bar{\Lambda}_0$. \blacksquare

Lemma 3.2 *Suppose that the conditions (B_1) – (B_3) and (F_1) – (F_2) hold with $p = 2$. If*

$$l_0 d_0 < \left(1 - \frac{1}{\beta_{k_0^*}^0}\right) < \frac{l_\infty}{d_*},$$

then there exists $\tilde{\Lambda}_0 > 0$ such that the conclusions of Lemma 3.1 hold for $\lambda > \tilde{\Lambda}_0$, where d_ is given by (3.3).*

Proof. Similarly as in the proof of Lemma 3.1, we can show that there exists $\Lambda_2^* > \Lambda_{k_0^*}^*$ such that

$$\inf_{(\mathcal{M}_\lambda^{k_0^*-1})^\perp \cap \mathbb{B}_{\lambda, \tilde{\rho}_0}} \mathcal{E}_\lambda(u) \geq \tilde{\kappa}_0$$

for all $\lambda > \Lambda_2^*$ with some $\tilde{\kappa}_0 > 0$ and $\tilde{\rho}_0 > 0$ independent of λ . In what follows, we will prove that there exists a positive constant $\tilde{R}_0 (> \tilde{\rho}_0)$ so large that

$$\sup_{\partial \mathcal{Q}_{\lambda, \tilde{R}_0}} \mathcal{E}_\lambda(u) \leq 0.$$

for λ sufficient large. In fact, if $u_\lambda \in \partial \mathcal{Q}_{\lambda, R}$ is such that $u_\lambda = R\tilde{u}$ with $\tilde{u}_\lambda \in \mathcal{Q}_{\lambda,1}$ then one of the following two cases must happen:

- (1) $\tilde{u}_\lambda \in \bigoplus_{i=1}^{k_0^*-1} \mathcal{N}_{\lambda,i}$ and $\|\tilde{u}_\lambda\|_\lambda \leq 1$;
(2) $\tilde{u}_\lambda \in \mathcal{Q}_{\lambda,1} \setminus \bigoplus_{i=1}^{k_0^*-1} \mathcal{N}_{\lambda,i}$ and $\|\tilde{u}_\lambda\|_\lambda = 1$.

In the case (1), it follows from Lemma 2.6 and the condition (F_3) that $\mathcal{E}_\lambda(R\tilde{u}_\lambda) \leq 0$ for all $R \geq 0$ and $\lambda > \Lambda_2^*$. In the case (2), by similar arguments as used in the proofs of (3.1) and Lemma 2.5, we have

$$\mathcal{E}_\lambda(R\tilde{u}) \leq R^2 \left(\frac{1}{2} \left(1 - \frac{1}{\beta_{k_0^*}} \right) + o_\lambda(1) - \int_{\mathbb{R}^N} \frac{F(x, R\tilde{u}_\lambda)}{(R\tilde{u}_\lambda)^2} \tilde{u}_\lambda^2 dx \right) \quad (3.4)$$

and

$$\int_{\mathbb{R}^N} \frac{F(x, R\tilde{u}_\lambda)}{(R\tilde{u}_\lambda)^2} \tilde{u}_\lambda^2 dx = \int_{\mathbb{R}^N} \frac{F(x, R\tilde{u})}{(R\tilde{u})^2} \tilde{u}^2 dx + o_\lambda(1) \quad (3.5)$$

for some $\tilde{u} \in \bigoplus_{i=1}^{k_0^*} \mathcal{N}_i$ with

$$\int_{\Omega} (|\Delta \tilde{u}|^2 + \max\{a_0, 0\} |\nabla \tilde{u}|^2 + \max\{b_0, 0\} |\tilde{u}|^2) dx = 1.$$

Thanks to the Fatou lemma, we deduce from the condition (F_2) and (3.3) that

$$\lim_{R \rightarrow +\infty} \int_{\mathbb{R}^N} \frac{F(x, R\tilde{u})}{(R\tilde{u})^2} \tilde{u}^2 dx \geq \int_{\mathbb{R}^N} \lim_{R \rightarrow +\infty} \frac{F(x, R\tilde{u})}{R^2 \tilde{u}^2} \tilde{u}^2 dx = \frac{l_\infty}{2} \|\tilde{u}\|_{L^2(\mathbb{R}^N)}^2 \geq \frac{l_\infty}{2d_*}.$$

Since $\frac{l_\infty}{d_*} > 1 - \frac{1}{\beta_{k_0^*}}$, by (3.4) and (3.5), there exist $\tilde{\Lambda}_0 \geq \Lambda_2^*$ and $\tilde{R}_0 > \tilde{\rho}_0$ independent of λ such that $\sup_{\partial \mathcal{Q}_{\lambda, R_0}} \mathcal{E}_\lambda(u) \leq 0$ for all $\lambda > \tilde{\Lambda}_0$. \blacksquare

By Lemmas 3.1 and 3.2, we know that $\mathcal{E}_\lambda(u)$ has a linking structure in E_λ for all $\lambda > \max\{\bar{\Lambda}_0, \tilde{\Lambda}_0\}$ in the case of $\min\{a_0, b_0\} < 0$. By the well known linking theorem, $\mathcal{E}_\lambda(u)$ has a Cerami sequence at level c_λ ($(C)_{c_\lambda}$ sequence for short) for all $\lambda > \max\{\bar{\Lambda}_0, \tilde{\Lambda}_0\}$, that is, there exists $\{u_{\lambda,n}\} \subset E_\lambda$ with $\lambda > \max\{\bar{\Lambda}_0, \tilde{\Lambda}_0\}$ such that

$$\mathcal{E}_\lambda(u_{\lambda,n}) = c_\lambda + o_n(1) \quad \text{and} \quad (1 + \|u_{\lambda,n}\|_\lambda) \mathcal{E}'_\lambda(u_{\lambda,n}) = o_n(1) \quad \text{strongly in } E_\lambda^*.$$

In the special case $k_0^* = 1$, the linking structure is actually the mountain pass geometry and the linking theorem can be replaced by the well known mountain pass theorem. Moreover, due to the conditions (F_3) and (F_4) , we can see that $c_\lambda \in [\min\{\kappa_0, \tilde{\kappa}_0\}, \frac{1}{2} \max\{R_0^2, \tilde{R}_0^2\}]$.

We next consider the case of $\min\{a_0, b_0\} \geq 0$.

Lemma 3.3 *Suppose that the conditions (F_1) – (F_2) hold and $\min\{a_0, b_0\} \geq 0$. Then there exists $l_\infty^* > 0$ such that*

$$\inf_{\mathbb{B}_{\rho_0^*}^*} \mathcal{E}_\lambda(u) \geq \kappa_0^* \quad \text{and} \quad \mathcal{E}_\lambda(R_0^* \phi_1) \leq 0$$

hold for $\lambda > \max\{0, -\frac{b_0}{b_\infty}\}$ with some $R_0^ > \rho_0^* > 0$ and $\kappa_0^* > 0$ independent of λ in the following two cases:*

- (a) $p > 2$;
(b) $p = 2$ and $l_\infty > l_\infty^*$.

Proof. Since $\mathcal{D}_\lambda(u, v) = \langle u, v \rangle_\lambda$ for all $(u, v) \in E_\lambda$ and $\mathcal{D}_\lambda(u, u)$ is definite on E_λ for $\lambda > \max\{0, -\frac{b_0}{b_\infty}\}$ in the case of $\min\{a_0, b_0\} \geq 0$, we can get the conclusions by similar but more simple arguments used in the proofs of Lemmas 3.1 and 3.2. \blacksquare

Due to Lemma 3.3, we can see that $\mathcal{E}_\lambda(u)$ has a mountain pass geometry for $\lambda > \max\{0, -\frac{b_0}{b_\infty}\}$ in the case of $\min\{a_0, b_0\} \geq 0$. It follows from the well known mountain pass theorem, $\mathcal{E}_\lambda(u)$ has a $(C)_{c_\lambda}$ sequence for all $\lambda > \max\{0, -\frac{b_0}{b_\infty}\}$. Furthermore, $c_\lambda \in [\kappa_0^*, \frac{1}{2}(R_0^*)^2(\bar{\mu}_1^2 + a_0 \bar{\mu}_1 + b_0)]$. In another word, there exists $\bar{\Lambda}_{*,0} > 0$ such that $\mathcal{E}_\lambda(u)$ always has a $(C)_{c_\lambda}$ sequence for $\lambda > \bar{\Lambda}_{*,0}$ and $c_\lambda \in [C, C']$ in both cases of $\min\{a_0, b_0\} \geq 0$ and $\min\{a_0, b_0\} < 0$.

Lemma 3.4 *Suppose that the conditions (B_1) – (B_3) and (F_1) – (F_2) and (F_4) hold with $p > 2$. Then there exist $\bar{\Lambda}_1 > \bar{\Lambda}_{*,0}$ and $C_0 > 0$ independent of λ such that $\|u_{\lambda,n}\|_\lambda \leq C_0 + o_n(1)$ for all $\lambda > \bar{\Lambda}_1$.*

Proof. By the condition (F_4) , we have

$$\begin{aligned} o_n(1) + c_\lambda &\geq \mathcal{E}_\lambda(u_{\lambda,n}) - \frac{1}{2} \langle \mathcal{E}_\lambda(u_{\lambda,n}), u_{\lambda,n} \rangle_{E_\lambda^*, E_\lambda} \\ &= \frac{1}{2} \int_{\mathbb{R}^N} (f(u_{\lambda,n})u_{\lambda,n} - 2F(u_{\lambda,n})) dx \\ &\geq \frac{l^*}{2} \|u_{\lambda,n}\|_{L^p(\mathbb{R}^N)}^p. \end{aligned}$$

On the other hand, due to the conditions (B_1) – (B_2) , for all $\lambda > \bar{\Lambda}_{*,0}$, we get from the Hölder and the Gagliardo-Nirenberg inequalities that

$$\begin{aligned} \mathcal{G}_\lambda(u_{\lambda,n}, u_{\lambda,n}) &\leq \max\{-a_0, 0\} B_0^2 \|\Delta u_{\lambda,n}\|_{L^2(\mathbb{R}^N)} \|u_{\lambda,n}\|_{L^2(\mathbb{R}^N)} \\ &\quad + \max\{-b_0, 0\} |\mathcal{B}_\infty|^{\frac{p-2}{p}} \|u_{\lambda,n}\|_{L^p(\mathbb{R}^N)}^2 \\ &\leq \max\{-a_0, 0\} B_0^2 |\mathcal{B}_\infty|^{\frac{p-2}{2p}} \|\Delta u_{\lambda,n}\|_{L^2(\mathbb{R}^N)} \|u_{\lambda,n}\|_{L^p(\mathbb{R}^N)} \\ &\quad + \max\{-a_0, 0\} B_0^2 \left(\frac{1}{\lambda b_\infty + b_0}\right)^{\frac{1}{2}} \|u_{\lambda,n}\|_\lambda^2 + \max\{-b_0, 0\} |\mathcal{B}_\infty|^{\frac{p-2}{p}} \|u_{\lambda,n}\|_{L^p(\mathbb{R}^N)}^2 \\ &\leq \left(\frac{1}{2} + \max\{-a_0, 0\} B_0^2 \left(\frac{1}{\lambda b_\infty + b_0}\right)^{\frac{1}{2}}\right) \|u_{\lambda,n}\|_\lambda^2 \\ &\quad + (2 \max\{-a_0, 0\}^2 B_0^4 + \max\{-b_0, 0\}) |\mathcal{B}_\infty|^{\frac{p-2}{p}} \|u_{\lambda,n}\|_{L^p(\mathbb{R}^N)}^2. \end{aligned}$$

It implies from the conditions (B_2) , (F_1) – (F_2) and the Hölder inequality that for all $\lambda > \bar{\Lambda}_{*,0}$,

$$\begin{aligned} \left| \int_{\mathbb{R}^N} F(u_{\lambda,n}) dx \right| &\leq 2l_0 \|u_{\lambda,n}\|_{L^2(\mathbb{R}^N)}^2 + C \|u_{\lambda,n}\|_{L^p(\mathbb{R}^N)}^p \\ &\leq \frac{2l_0}{\lambda b_\infty + b_0} \|u_{\lambda,n}\|_\lambda^2 + 2l_0 |\mathcal{B}_\infty|^{\frac{p-2}{p}} \|u_{\lambda,n}\|_{L^p(\mathbb{R}^N)}^2 + C \|u_{\lambda,n}\|_{L^p(\mathbb{R}^N)}^p. \end{aligned}$$

Now, we can see from $\mathcal{E}_\lambda(u_{\lambda,n}) = c_\lambda + o_n(1)$ that

$$\left(\frac{1}{2} + o_\lambda(1)\right) \|u_{\lambda,n}\|_\lambda^2 \leq C' c_\lambda + o_n(1).$$

Note that $c_\lambda \in [C, C']$, there exist $\bar{\Lambda}_1 > \bar{\Lambda}_{*,0}$ and $C_0 > 0$ independent of λ such that $\|u_{\lambda,n}\|_\lambda \leq C_0 + o_n(1)$ for all $\lambda > \bar{\Lambda}_1$. \blacksquare

Lemma 3.5 *Suppose that the conditions (B_1) – (B_3) and (F_1) – (F_3) hold with $p = 2$. If $l_\infty \notin \sigma(\Delta^2 - a_0 \Delta + b_0, L^2(\Omega))$, then there exists $\tilde{\Lambda}_1 > \bar{\Lambda}_{*,0}$ such that $\{u_{\lambda,n}\}$ is bounded in E_λ for all $\lambda > \tilde{\Lambda}_1$.*

Proof. Suppose on the contrary that there exists a subsequence of $\{u_{\lambda,n}\}$, which is still denoted by $\{u_{\lambda,n}\}$, such that $\|u_{\lambda,n}\|_\lambda \rightarrow +\infty$ as $n \rightarrow +\infty$. Let $w_{\lambda,n} = \frac{u_{\lambda,n}}{\|u_{\lambda,n}\|_\lambda}$. Then without loss of generality, we may assume that $w_{\lambda,n} \rightharpoonup w_{\lambda,0}$ weakly in E_λ for some $w_{\lambda,0} \in E_\lambda$ as $n \rightarrow \infty$.

Claim 1: There exists $\Lambda_3^* > \bar{\Lambda}_{*,0}$ such that $w_{\lambda,0} \neq 0$.

Indeed, if $w_{\lambda,0} = 0$, then by Remark 2.1, we can see that $w_{\lambda,n}^- \rightarrow 0$ strongly in E_λ with $\lambda > \bar{\Lambda}_{*,0}$ as $n \rightarrow \infty$, where $w_{\lambda,n}^-$ is the projection of $w_{\lambda,n}$ in $\bigoplus_{i=1}^{k_0^*} \mathcal{N}_{\lambda,i}$. It follows from Lemma 2.6 that

$$\mathcal{D}_\lambda(w_{\lambda,n}, w_{\lambda,n}) \geq \left(1 - \frac{1}{\beta_{k_0^*}^0} + o_\lambda(1)\right) \|w_{\lambda,n}^+\|_\lambda^2 + o_n(1), \quad (3.6)$$

where $w_{\lambda,n}^+ = w_{\lambda,n} - w_{\lambda,n}^-$. On the other hand, thanks to the conditions (F_2) – (F_3) , we see from $(1 + \|u_{\lambda,n}\|_\lambda) \mathcal{E}'_\lambda(u_{\lambda,n}) = o_n(1)$ strongly in E_λ^* that

$$\mathcal{D}_\lambda(w_{\lambda,n}, w_{\lambda,n}) = o_n(1) + \int_{\mathbb{R}^N} \frac{f(u_{\lambda,n})}{u_{\lambda,n}} w_{\lambda,n}^2 dx \leq o_n(1) + l_\infty \|w_{\lambda,n}\|_{L^2(\mathbb{R}^N)}^2,$$

which, together with the Sobolev embedding theorem, the fact that E_λ is continuously embedded into $H^2(\mathbb{R}^N)$ for $\lambda > \max\{0, \frac{-b_0}{b_\infty}\}$ and the condition (B_2) , implies that

$$\mathcal{D}_\lambda(w_{\lambda,n}, w_{\lambda,n}) \leq \frac{4l_\infty}{\lambda b_\infty + b_0} \|w_{\lambda,n}^+\|_\lambda^2 + o_n(1). \quad (3.7)$$

Thanks to Lemma 2.5, we can deduce from (3.6) and (3.7) that there exists $\Lambda_3^* > \bar{\Lambda}_{*,0}$ such that $w_{\lambda,n}^+ \rightarrow 0$ strongly in E_λ as $n \rightarrow \infty$ for $\lambda > \Lambda_3^*$, which is inconsistent with $\|w_{\lambda,n}\|_\lambda = 1$ for all $n \in \mathbb{N}$.

Claim 2: There exists $\Lambda_4^* > \Lambda_3^*$ such that $w_{\lambda,n} \rightarrow w_{\lambda,0}$ strongly in E_λ as $n \rightarrow \infty$ for $\lambda > \Lambda_4^*$ up to a subsequence.

In fact, let $\mathcal{Q}_{\lambda,0} = \{x \in \mathbb{R}^N : w_{\lambda,0} \neq 0\}$, then $|u_{\lambda,n}| \rightarrow +\infty$ as $n \rightarrow \infty$ on $\mathcal{Q}_{\lambda,0}$. It follows from the conditions (F_2) – (F_3) and a variant of the Lebesgue dominated convergence theorem (cf. [22, Theorem 2.2]) that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \frac{f(u_{\lambda,n})v}{\|u_{\lambda,n}\|_\lambda} dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \frac{f(u_{\lambda,n})}{u_{\lambda,n}} w_{\lambda,n} v \chi_{\mathcal{Q}_{\lambda,0}} dx = \int_{\mathbb{R}^N} l_\infty w_{\lambda,0} v dx$$

for every $v \in H^2(\mathbb{R}^N)$. Since $w_{\lambda,n} \rightharpoonup w_{\lambda,0}$ weakly in E_λ as $n \rightarrow \infty$, due to the fact that E_λ is continuously embedded into $H^2(\mathbb{R}^N)$ for $\lambda > \max\{0, \frac{-b_0}{b_\infty}\}$, we have that $\lim_{n \rightarrow \infty} \mathcal{D}_\lambda(w_{\lambda,n}, v) = \mathcal{D}_\lambda(w_{\lambda,0}, v)$ for all $v \in H^2(\mathbb{R}^N)$. Thus, $w_{\lambda,0} \in H^2(\mathbb{R}^N)$ satisfies the following equation in the weak sense:

$$\Delta^2 w_{\lambda,0} - a_0 \Delta w_{\lambda,0} + (\lambda b(x) + b_0) w_{\lambda,0} = l_\infty w_{\lambda,0}, \quad \text{in } \mathbb{R}^N. \quad (3.8)$$

Let $I_\lambda(u) = \frac{1}{2}(\mathcal{D}_\lambda(u, u) - l_\infty \|u\|_{L^2(\mathbb{R}^N)}^2)$. Then $I'_\lambda(w_{\lambda,0}) = 0$ in E_λ^* . Now, by Remark 2.1 and a similar argument used in the proof of (2.11), we have

$$\begin{aligned} o_n(1) &= \left\langle \frac{\mathcal{E}'_\lambda(u_{\lambda,n})}{\|u_{\lambda,n}\|_\lambda} - I'_\lambda(w_{\lambda,0}), w_{\lambda,n} - w_{\lambda,0} \right\rangle_{E_\lambda^*, E_\lambda} \\ &\geq \left(1 - \frac{1}{\beta_{k_0^*}^0} + o_\lambda(1)\right) \|w_{\lambda,n} - w_{\lambda,0}\|_\lambda^2 + o_n(1) - \int_{\mathbb{R}^N} \frac{f(u_{\lambda,n})}{u_{\lambda,n}} |w_{\lambda,n} - w_{\lambda,0}|^2 dx \\ &\quad - \int_{\mathbb{R}^N} \left(\frac{f(u_{\lambda,n})}{u_{\lambda,n}} - l_\infty\right) (w_{\lambda,n} - w_{\lambda,0}) w_{\lambda,0} dx. \end{aligned} \quad (3.9)$$

Note that $(w_{\lambda,n} - w_{\lambda,0}) w_{\lambda,0} = o_n(1)$ strongly in $L^1(\mathbb{R}^N)$. By the conditions (F_2) – (F_3) , the Sobolev embedding theorem and a variant of the Lebesgue dominated convergence theorem (cf. [22, Theorem 2.2]), we can see that

$$\begin{aligned} &\int_{\mathbb{R}^N} \frac{f(u_{\lambda,n})}{u_{\lambda,n}} |w_{\lambda,n} - w_{\lambda,0}|^2 dx + \int_{\mathbb{R}^N} \left(\frac{f(u_{\lambda,n})}{u_{\lambda,n}} - l_\infty\right) (w_{\lambda,n} - w_{\lambda,0}) w_{\lambda,0} dx \\ &\leq \frac{l_\infty}{\lambda b_\infty + b_0} \|w_{\lambda,n} - w_{\lambda,0}\|_\lambda^2 + o_n(1). \end{aligned} \quad (3.10)$$

Combining (3.9) and (3.10), we must have that there exists $\Lambda_4^* > \Lambda_3^*$ such that $w_{\lambda,n} \rightarrow w_{\lambda,0}$ strongly in E_λ as $n \rightarrow \infty$ for $\lambda > \Lambda_4^*$.

Claim 3: $w_{\lambda,0} \rightarrow w_{\infty,0}$ strongly in $H^1(\mathbb{R}^N)$ as $\lambda \rightarrow +\infty$ up to a subsequence for some $w_{\infty,0} \in H$ which satisfies the following equation in the weak sense:

$$\Delta^2 w_{\infty,0} - a_0 \Delta w_{\infty,0} + b_0 w_{\infty,0} = l_\infty w_{\infty,0}, \quad \text{in } \Omega. \quad (3.11)$$

Indeed, since $\|w_{\lambda,n}\|_\lambda = 1$, by (2.4) and (2.5) and a similar argument used in the proof of Lemma 2.1, we can show that $w_{\lambda,0} \rightharpoonup w_{\infty,0}$ weakly in $H^2(\mathbb{R}^N)$ and $w_{\lambda,0} \rightarrow w_{\infty,0}$ strongly in $H^1(\mathbb{R}^N)$ for some $w_{\infty,0} \in H$ with $w_{\infty,0} = 0$ outside Ω as $\lambda \rightarrow +\infty$, up to a subsequence. It follows from (3.8) that $w_{\infty,0} \in H$ satisfies (3.11) in the weak sense.

Now, multiplying respectively (3.11) and (3.8) with $w_{\infty,0}$ and $w_{\lambda,0}$, and integrating, we can see that

$$\begin{aligned} & \|\Delta w_{\infty,0}\|_{L^2(\mathbb{R}^N)}^2 + a_0 \|\nabla w_{\infty,0}\|_{L^2(\mathbb{R}^N)}^2 + b_0 \|w_{\infty,0}\|_{L^2(\mathbb{R}^N)}^2 \\ & \leq \lim_{\lambda \rightarrow +\infty} (\|w_{\lambda,0}\|_\lambda^2 + \mathcal{G}_\lambda(w_{\lambda,0}, w_{\lambda,0})) \\ & = l_\infty \lim_{\lambda \rightarrow +\infty} \|w_{\lambda,0}\|_{L^2(\mathbb{R}^N)}^2 \\ & = l_\infty \|w_{\infty,0}\|_{L^2(\mathbb{R}^N)}^2 \\ & = \|\Delta w_{\infty,0}\|_{L^2(\mathbb{R}^N)}^2 + a_0 \|\nabla w_{\infty,0}\|_{L^2(\mathbb{R}^N)}^2 + b_0 \|w_{\infty,0}\|_{L^2(\mathbb{R}^N)}^2. \end{aligned}$$

Hence, $\int_{\mathbb{R}^N} \lambda b(x) w_{\lambda,0}^2 dx = o_\lambda(1)$ and $w_{\lambda,0} \rightarrow w_{\infty,0}$ strongly in $H^2(\mathbb{R}^N)$ as $\lambda \rightarrow +\infty$ up to a subsequence. Therefore, by Claim 1 and Claim 2, $\|w_{\infty,0}\|_{\Omega,0} = 1$, which gives $w_{\infty,0} \neq 0$. It follows from Claim 2 that $l_\infty \in \sigma(\Delta^2 - a_0 \Delta + b_0, L^2(\Omega))$, which contradicts the assumption that $l_\infty \notin \sigma(\Delta^2 - a_0 \Delta + b_0, L^2(\Omega))$. Thus, there exist $\tilde{\Lambda}_1 > \bar{\Lambda}_{*,0}$ such that $\{u_{\lambda,n}\}$ is bounded in E_λ for all $\lambda > \tilde{\Lambda}_1$. \blacksquare

Remark 3.1 Since $\text{span}\{\phi_k\} = H$ and ϕ_k are orthogonal in H , it is easy to show that $\sigma(\Delta^2 - a_0 \Delta + b_0, L^2(\Omega)) = \{\mu_k^2 + a_0 \mu_k + b_0\}$.

Now, we can give the proofs of Theorems 1.1-1.3.

Proof of Theorem 1.1: By Lemma 3.4, $u_{\lambda,n} \rightharpoonup u_{\lambda,0}$ weakly in E_λ with $\lambda > \bar{\Lambda}_1$ as $n \rightarrow \infty$ up to a subsequence. Without loss of generality, we may assume that $u_{\lambda,n} \rightharpoonup u_{\lambda,0}$ weakly in E_λ with $\lambda > \bar{\Lambda}_1$ as $n \rightarrow \infty$. Since $\mathcal{E}_\lambda(u)$ is C^1 , it is easy to see from the fact that $\{u_{\lambda,n}\}$ is a $(C)_{c_\lambda}$ sequence that $\mathcal{E}'_\lambda(u_{\lambda,0}) = 0$ in E_λ^* with $\lambda > \bar{\Lambda}_1$. It remains to show that $u_{\lambda,0} \neq 0$ in E_λ for λ sufficiently large. Indeed, if $u_{\lambda,0} = 0$, then by the conditions (B_1) – (B_2) and (F_1) – (F_2) and the the Sobolev, the Hölder, the Gagliardo-Nirenberg inequalities and the fact that E_λ is embedded continuously into $H^2(\mathbb{R}^N)$ for $\lambda > \max\{0, \frac{-b_0}{b_\infty}\}$ that

$$\begin{aligned} \mathcal{G}_\lambda(u_{\lambda,n}, u_{\lambda,n}) & \leq \max\{-a_0, 0\} B_0^2 \|\Delta u_{\lambda,n}\|_{L^2(\mathbb{R}^N)} \|u_{\lambda,n}\|_{L^2(\mathbb{R}^N)} + o_n(1) \\ & \leq \max\{-a_0, 0\} B_0^2 \left(\left(\frac{1}{\lambda b_\infty + b_0} \right)^{\frac{1}{2}} + o_n(1) \right) \|u_{\lambda,n}\|_\lambda^2 + o_n(1) \end{aligned}$$

and

$$\begin{aligned} \left| \int_{\mathbb{R}^N} F(u_{\lambda,n}) dx \right| & \leq 2l_0 \|u_{\lambda,n}\|_{L^2(\mathbb{R}^N)}^2 + C \|u_{\lambda,n}\|_{L^p(\mathbb{R}^N)}^p \\ & \leq 2l_0 \left(\left(\frac{1}{\lambda b_\infty + b_0} \right)^{\frac{1}{2}} \|u_{\lambda,n}\|_\lambda^2 + o_n(1) \right) + C \|\Delta u_{\lambda,n}\|_{L^2(\mathbb{R}^N)}^{\frac{p}{2}} \|u_{\lambda,n}\|_{L^2(\mathbb{R}^N)}^{\frac{p}{2}} \\ & \leq 2l_0 \left(\left(\frac{1}{\lambda b_\infty + b_0} \right)^{\frac{1}{2}} \|u_{\lambda,n}\|_\lambda^2 + o_n(1) \right) + C \left(\left(\frac{1}{\lambda b_\infty + b_0} \right)^{\frac{p}{2}} + o_n(1) \right) \|u_{\lambda,n}\|_\lambda^p, \end{aligned}$$

which, together with $\langle \mathcal{E}_\lambda(u_{\lambda,n}), u_{\lambda,n} \rangle_{E_\lambda^*, E_\lambda} = o_n(1)$ and Lemma 3.4, yields that there exists $\hat{\Lambda} > \bar{\Lambda}_1$ such that $u_{\lambda,n} \rightarrow 0$ strongly in E_λ with $\lambda > \hat{\Lambda}$ as $n \rightarrow \infty$. It follows that $c_\lambda = 0$ for $\lambda > \hat{\Lambda}$. It is impossible since $c_\lambda \geq C > 0$ for all $\lambda > \bar{\Lambda}_1$. \blacksquare

Proof of Theorem 1.2: If we can show that $\{u_{\lambda,n}\}$ is uniformly bounded in E_λ as Lemma 3.4 in this case, that is, there exist $\tilde{\Lambda}_2 > \tilde{\Lambda}_1$ and $C_0 > 0$ independent of $\lambda > \tilde{\Lambda}_2$ such that $\|u_{\lambda,n}\|_\lambda \leq C_0 + o_n(1)$ with $\lambda > \tilde{\Lambda}_2$, then we can follow the proof of Theorem 1.1 to obtain the conclusion.

In fact, by Lemma 3.5, there exists $C_\lambda > 0$ such that $\|u_{\lambda,n}\|_\lambda \leq C_\lambda$ for all $n \in \mathbb{N}$ with $\lambda > \tilde{\Lambda}_1$. If $C_\lambda \rightarrow +\infty$ as $\lambda \rightarrow +\infty$ up to a subsequence, then there exists $\lambda_m \rightarrow +\infty$ as $m \rightarrow \infty$ and $n_m \in \mathbb{N}$ such that $\|u_{\lambda_m,n_m}\|_{\lambda_m} \rightarrow +\infty$ as $m \rightarrow \infty$. Let $w_m = \frac{u_{\lambda_m,n_m}}{\|u_{\lambda_m,n_m}\|_{\lambda_m}}$, then without loss of generality, we may assume that $w_m \rightharpoonup w_0$ in $H^2(\mathbb{R}^N)$ as $m \rightarrow \infty$ for some $w_0 \in H^2(\mathbb{R}^N)$ due to (2.4) and (2.5). Now, by using similar arguments in the proof of Lemma 3.5, we can show that w_0 is a nontrivial weak solution of (3.11), which is inconsistent with the assumption that $l_\infty \notin \sigma(\Delta^2 - a_0\Delta + b_0, L^2(\Omega))$. ■

Proof of Theorem 1.3: Suppose that u_λ is the nontrivial solution of (\mathcal{P}_λ) obtained by Theorem 1.1 or Theorem 1.2 with λ large enough. We can see from Lemma 3.4 and the proof of Theorem 1.2 that $\|u_\lambda\|_\lambda \leq C_0$ for all λ with some $C_0 > 0$ independent of λ . Now, similarly as in the proof of Lemma 3.5, we can show that $u_\lambda \rightarrow u_*$ strongly in $H^2(\mathbb{R}^N)$ as $\lambda \rightarrow +\infty$ for some $u_* \in H$ with $u_* \equiv 0$ outside Ω , up to a subsequence. Furthermore, we also have $\lambda \int_{\mathbb{R}^N} b(x)u_\lambda^2 dx = o_\lambda(1)$ and $\mathcal{F}'(u_*) = 0$ in H^* , where H^* is the dual space of H and

$$\mathcal{F}(u) = \frac{1}{2} \int_{\Omega} (|\Delta u|^2 + a_0 |\nabla u|^2 + b_0 u^2) dx - \int_{\Omega} F(u) dx.$$

Note that $\mathcal{E}_\lambda(u_\lambda) = c_\lambda \geq C > 0$ for all $\lambda > \max\{\bar{\Lambda}_1, \tilde{\Lambda}_1\}$, we must have $u_* \neq 0$ in H . Thus, u_* is a nontrivial weak solution of (1.3). ■

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